Stochastic trace formula for closed, negatively curved manifolds

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My *1997 Ph.D. thesis* as a blog entry.

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6 Notation

M is a negatively curved dim $= n$ closed Riemannian manifold with metric g, metric connection ∇, and (nonnegative) Laplace-Beltrami Operator Δ_M. Let $k_{-t\Delta/2}(x, y)$ represent the heat kernel on M.

Hence $k_{-t\Delta/2}(x, x) = dDM_*\mu/\sqrt{g}dx$ is the Radon-Nicodym derivative of n-dimensional Wiener Measure μ, restricted to the pull-back of continuous loop space $\Omega_t(M)|_x$, via the inverse of the Weiner measure-preserving development map DM. *Note:* $DM^{-1}\Omega_t|_x$ is not a loop space.

Ω_t^0 is the space of continuous contractible loops on M.

$\Omega_t[\gamma]$ is the space of continuous loops on M homotopic to the closed geodesic γ. Let γ_0 be its primitive loop.

$DM^{-1}\Omega_t^0[\gamma]$ is the preimage of continuous contractible loops on M written as offsets homotopic to $\gamma(s) = DM(\frac{s\ell(\gamma)}{t}\vec{e}^1), 0 \le s \le t$. Think Horocyclic Coordinates – each fiber as the geometric limit of periodic geodesic spheres $S^{n-1}_{\gamma_0(s)}(k\ell(\gamma_0)), 0 \le s \le t, k \to \infty$, vectorized in the Normal Bundle over γ_0. Our curvature constraints imply Horocyclic Coordinates for every γ_0 exist as a smooth, DM-compatible coordinate map for $\Omega_t^0[\gamma]$.

Now $\vec{x}(\tau)+\ell(\gamma)\vec{e}^1$ is the *undeveloped* endpoint of the "offset" *kinked geodesic* homotopic to $\gamma : DM(\vec{x}(\tau)+\frac{s\ell(\gamma)}{t}\vec{e}^1), 0 \le s \le t$. The curve is periodic with period $\ell(\gamma_0)$, and it revisits its kinked starting point $DM(\vec{x}(\tau))$ at time t, making the computation of its forward derivative $J = \lim_{s\uparrow t} DM'|_{DM(\vec{x}(\tau)+\frac{\ell(\gamma)s}{t}\vec{e}^1)}$ tractible as a linear automorphism of $T_{DM(\vec{x}(\tau))}M$. Importantly, $J_{DM(\vec{x}(\tau)+\ell(\gamma)\vec{e}^1)}$ may be constructed using **Jacobi Fields**, since DM **is** the (iterated) exponential map along any series of connected straight lines in $\mathbb{R}^n$. We will study $1/2\int_0^t \langle dX|dX\rangle_s$, with the solution

$$X_t = X_0 + \int_0^t \sqrt{J}_{X_t} dB_t \tag{1}$$

$Z_{-\Delta/2}(t) := \int_M k_{-t\Delta/2}(x,x)\sqrt{g}dx = \sum_{j=0}^{\infty} e^{-\lambda_i t/2}$ is the trace of the heat kernel.

Finally let us define the following from their Radon-Nicodym derivatives:

$$\begin{aligned} DM_*\mu(\Omega_t) &:= \int_M DM_*\mu(\Omega_t|_x\sqrt{g}dx) \\ DM_*\mu(\Omega_t^0) &:= \int_M DM_*\mu(\Omega_t^0|_x\sqrt{g}dx) \\ DM_*\mu(\Omega_t[\gamma]) &:= \int_M DM_*\mu(\Omega_t[\gamma]|_x\sqrt{g}dx) \end{aligned} \tag{2}$$

7 Stochastic Trace Formula

$$Z_{-\Delta/2}(t) = DM_*\mu(\Omega_t) = DM_*\mu(\Omega_t^0) + \sum_{\{\gamma\}} DM_*\mu(\Omega_t[\gamma])$$

$$DM_*\mu(\Omega_t^0) \approx_{t\to 0} (2\pi t)^{-n/2}(vol(M) + t/6\int_M K(x)\sqrt{g}dx + O(t^2))\text{by McKean-Singer}$$

$$\begin{aligned} DM_*\mu(\Omega_t[\gamma]) &= e^{-\ell(\gamma)^2/2t}\int_M DM_*\mu(e_t^{\langle J_B B_t|B_t\rangle}\Omega_t^0[\gamma]|_x\sqrt{g}dx) \text{ by Cameron-Martin} \\ &= e^{-\ell(\gamma)^2/2t}\int_{T_{\gamma_0}M} E(e_t^{J_B}|\Omega_t^0[\gamma]|_{x(\tau)})dx^1(\tau)\dots dx^n(\tau)d\tau \end{aligned}$$

$$\frac{dDM_*\mu(e^{-\ell(\gamma)x^1(t)}\Omega_t^0[\gamma])}{dx^1(\tau)\dots dx^n(\tau)d\tau}|_{y\vec{(}\tau)} \approx_{t\to 0} \frac{e^{-\left\langle |I-J_{DM(\vec{x}(\tau),\vec{y}(\tau))}\vec{x}(\tau)\middle|\vec{x}(\tau)\right\rangle/2t}}{(2\pi t)^{(n+1)/2}}(1+O(t^2)) \text{ semi-classical limit}$$

$$\text{Horocyclic coordinates} : z(\tau) - x(\tau) = x + \ell(\gamma)\vec{e}^1 \implies$$

$$\begin{aligned} \int_{M/S^1\oplus S^1} k_t(x,z)dx &= \lim_{j\to\infty}\frac{e^{-\ell(\gamma)^2/2t}}{\sqrt{2\pi t}}E(e^{\left\langle J_{X_t^j}\vec{x}\middle|\vec{x}\right\rangle}) \\ &= \lim_{j\to\infty}\frac{e^{-\ell(\gamma)^2/2t}}{\sqrt{2\pi t}}\int_{M^j/S^1\oplus S^1}\frac{1}{\sqrt{2\pi t}^{jn}\det|I-J_{X^j}|}e^{-\ell(X^j)^2/2t}X^j \end{aligned} \tag{3}$$

8 Approximation and the Selberg Trace Formula

In the dim $= 2$ constant curvature $-\kappa^2$ surface case,

$$
\begin{aligned}
\sqrt{J_{\vec{x},\vec{y}}}\,dRB &= \begin{pmatrix} e^{\kappa d(\vec{x},\vec{y})/2} & 0 \\ 0 & e^{-\kappa d(\vec{x},\vec{y})/2} \end{pmatrix} \implies \\
\left\langle \sqrt{J}dRB \middle| \sqrt{J}dRB \right\rangle &= e^{\kappa\ell(B)} dRB_1^2 - e^{-\kappa\ell(B)} dRB_2^2 \\
\int_0^t \left\langle \sqrt{J}dB \middle| \sqrt{J}dB \right\rangle &= e^{\kappa\ell(\gamma)} - e^{-\kappa\ell(\gamma)} \\
\det I - J_\gamma &= (e^{\kappa\ell(\gamma)/2} - e^{-\kappa\ell(\gamma)/2})^2
\end{aligned}
\tag{4}
$$

which is constant over $(\vec{x}, \tau)$, so the approximation $\approx_{t\to 0}$ line in Equation (2) becomes *exact*:

$$
\begin{aligned}
DM_*\mu(\Omega_t[\gamma]) &= \frac{e^{-\ell(\gamma)^2/2t}\ell(\gamma_0)}{\sqrt{2\pi t}(e^{\kappa\ell(\gamma)/2} - e^{-\kappa\ell(\gamma)/2})} \\
\gamma(t) = \gamma_0(kt) \implies & \\
&= \frac{e^{-k^2\ell(\gamma_0)^2/2t}\ell(\gamma_0)}{2\sqrt{2\pi t}\sinh k\kappa\ell(\gamma_0)/2}
\end{aligned}
\tag{5}
$$

In the dim $= 3$ hyperbolic manifold case, we use complex coordinates $(z, \bar{z})$ on the normal bundle to write

$$
\begin{aligned}
J_{DM(\vec{x}+(\tau+\ell(\gamma))\vec{e}^1)} &= \begin{pmatrix} e^{\kappa\ell(\gamma)} & 0 & 0 \\ 0 & e^{-\kappa\ell(\gamma)+i\theta(\gamma)} & 0 \\ 0 & 0 & e^{-\kappa\ell(\gamma)-i\theta(\gamma)} \end{pmatrix} \\
\implies & \\
\det I - {\perp_{\gamma_0}}^k &= |1 - e^{-k(\kappa\ell(\gamma_0)-i\theta(\gamma_0))}|^2
\end{aligned}
\tag{6}
$$

and since $z = x^2 + ix^3 \implies d\bar{z} \wedge dz = (dx^2 - idx^3) \wedge (dx^2 + idx^3) = 2idx^2 \wedge dx^3$, the approximation in Equation (2) again becomes exact:

$$
\begin{aligned}
\kappa = 1 \implies & \\
DM_*\mu(\Omega_t[\gamma]) &= \frac{e^{-k^2\ell(\gamma_0)^2/2t}\ell(\gamma_0)}{2\sqrt{2\pi t(1 - e^{-k\ell(\gamma_0)})}\,|e^{k\ell(\gamma_0)/2} - e^{-k(\ell(\gamma_0)/2 - i\theta(\gamma_0))}|}
\end{aligned}
\tag{7}
$$